CONCLUSION

Heat and mass transfer for the main equipment in absorption heat pumps have been discussed together with the design aspects for the absorbers.

The mass flow rate of the working fluid vapour has shown a significant effect on the liquid phase mass transfer coefficient and the cooling temperature in the evaporator. The cooling capacity of the absorption system has been shown to increase with increases in the mass flow rate of the aqueous lithium bromide solution used as an absorbent.

REFERENCES

1. D. B. Spalding, *Convective Mass Transfer, An Introduction*. Edward Arnold, London (1963).
2. S. Gabsi and R. Bugrel, Water absorption by a solution of lithium bromide, *Int. Commun. Heat Mass Transfer* **10,** 49 (1983).
3. J. E. Vivian and D. W. Peaceman, Liquid-side resistor in gas absorption, *J. Am Inst. Chem. Engrs* **2,** 437 (1956).
4. G. Astarita, *Mass Transfer with Chemical Reaction*. Elsevier, London (1967).
5. M. A. R. Eisa, S. Devotta and F. A. Holland, A study of economiser performance in a water–lithium bromide cooler, *Int. J. Heat Mass Transfer* **28,** 2323 (1985).
6. I. E. Smith, Heat pump absorbers, *Proc. NATO Advanced Study Institute on Heat Pump Fundamentals*, Espinho, Spain, Sept., pp. 1–12 (1980).
7. S. A. Bogatykh and I. D. Eaovich, A study of the viscosities of aqueous solutions of LiCl, LiBr and CaCl$_2$ applicable to the normal drying of gases, *J. Appl. Chem. U.S.S.R.* **36,** 180 (1965).
8. S. A. Bogatykh and I. D. Eaovich, Investigation of densities of aqueous solutions of LiCl, LiBr and CaCl$_2$ in relation to conditions of gas drying, *J. Appl. Chem. U.S.S.R.* **38,** 932 (1965).

Int. J. Heat Mass Transfer. Vol. 34, No. 3, pp. 894–897, 1991
Printed in Great Britain

0017-9310/91 $3.00+0.00
Pergamon Press plc

A suitable approximate solution of Neumann's problem

Didier Lecomte and Jean-Christophe Batsale
L.E.M.T.A., U.R.A. C.N.R.S. 875, Ecole des Mines, Parc de Saurupt, 54000 Nancy Cedex, France

(*Received* 15 *December* 1989 *and in final form* 9 *April* 1990)

1. INTRODUCTION

Moving boundary problems have few exact analytical solutions. The most famous one was expressed by Neumann in the previous century. We will briefly report his results [1].

The problem considers a semi-infinite body of phase-change material extending from $x = 0$ to ∞. The initial temperature T_i is assumed to be uniform and higher than the solidification temperature T_c. Dirichlet's condition is applied to the fixed boundary at a temperature T_0 ($T_0 < T_c$). Crystallization is observed and if one assumes only conductive heat transfer in both phases, the position of the boundary layer may be expressed by

$$s = 2K_N\sqrt{(a_1 t)}. \tag{1}$$

The temperature profiles in the solid and the liquid phases are

$$0 < x < s(t) \quad T_1(x,t) = T_0 + \frac{(T_c - T_0)}{\operatorname{erf}(K_N)}\operatorname{erf}\left(\frac{x}{2\sqrt{(a_1 t)}}\right) \tag{2}$$

$$x > s(t) \quad T_2(x,t) = T_i - \frac{(T_i - T_c)}{\operatorname{erfc}(K_N\alpha)}\operatorname{erfc}\left(\frac{x}{2\sqrt{(a_2 t)}}\right) \tag{3}$$

where K_N is a function of three dimensionless parameters:

$$Ste = \frac{\rho_1 C_1 (T_c - T_0)}{\rho_1 L}$$

Stefan number (ratio of the sensible heat in the solid phase to the latent heat released by crystallization);

$$\phi = \frac{\rho_2 C_2 (T_i - T_c)}{\rho_1 C_1 (T_c - T_0)}$$

ratio of the sensible heat in the liquid phase to the sensible heat in the solid phase;

$$\alpha = \sqrt{(a_1/a_2)}$$

square root of the ratio of the thermal diffusivities. The solution of the transcendent equation is K_N

$$\frac{e^{-K_N^2}}{\operatorname{erf}(K_N)} - \frac{e^{-K_N^2\alpha^2}}{\operatorname{erfc}(K_N\alpha)}\frac{\phi}{\alpha} = \frac{K_N\sqrt{\pi}}{Ste}. \tag{4}$$

Equation (4) can be solved numerically but the solution can be fairly difficult to obtain when the function erfc ($K_N\alpha$) is near zero.

For various boundary conditions or various geometries, it is useful to have approximate solutions. Several methods were developed for the case where the initial overheating is zero ($\phi = 0$). In this particular case, the thermal properties of the liquid phase do not appear in the solution.

The quasi-steady approximation [2] assumes a linear profile of the temperature in the solid region and is only valid for very small Stefan numbers. For larger Stefan numbers, higher order polynomial approximations have to be used. The coefficients are determined so as to satisfy the boundary conditions. The heat equation should also be satisfied either in its integral form [3] or at a number of discrete points [3–5].

For small but non-negligible Stefan numbers, perturbation methods also yield good results [6, 7].

To find approximate solutions for the case $\phi \neq 0$, it is necessary to make assumptions on temperature profiles in both phases [8–11]. The solution is seldom simple and it is often preferable to use numerical methods [12], which allow for instance, the practical choice of the initial conditions.

In the present paper, we intend to show how a correct choice of an approximate solution may yield accurate results. In the case of Neumann's problem, the approximate solution has a very simple form and may easily be compared to the exact one.

2. STATEMENT OF THE PROBLEM

The equations for Neumann's problem are well known.

- Solid phase

$$0 < x < s(t), \quad \frac{\partial T_1}{\partial t} = a_1 \frac{\partial^2 T_1}{\partial x_1^2}. \tag{5}$$

NOMENCLATURE

a	thermal diffusivity [$m^2 s^{-1}$]
C	specific heat [$J kg^{-1} K^{-1}$]
d	thermal wave penetration depth [m]
L	latent heat of crystallization [$J kg^{-1}$]
n	polynomial exponent (liquid phase)
O	first-order approximation function
s	crystallization front location [m]
Ste	Stefan number
t	time [s]
T	temperature [°C]
T_i	initial temperature [°C]
T_0	temperature at the fixed boundary [°C]
T_c	phase-change temperature [°C]
x	space coordinate variable [m].

Greek symbols

α	square root of diffusivities ratio
ε	relative error
λ	thermal conductivity [m K]
ρ	density [$kg m^{-3}$]
ϕ	sensible heat ratio.

Functions

erf	error function
erfc	complementary error function.

Subscripts

1	solid phase
2	liquid phase
A	present approximate solution
G	Goodman's approximate solution [3]
N	Neumann's exact solution [1]
S	Solomon's approximate solution [12].

- Liquid phase

$$s(t) < \infty, \quad \frac{\partial T_2}{\partial t} = a_2 \frac{\partial^2 T_2}{\partial x_2^2}. \tag{6}$$

The boundary conditions are, for both phases:

$$x = s(t), \quad -\lambda_2 \frac{\partial T_2}{\partial x}\bigg|_{x=s^+} + \lambda_1 \frac{\partial T_1}{\partial x}\bigg|_{x=s^-} = \rho_1 L\dot{s} \tag{7}$$

$$T_1|_{x=s^-} = T_c \tag{8}$$

$$T_2|_{x=s^+} = T_c \tag{9}$$

$$x = 0, \quad T_1|_{x=0} = T_0. \tag{10}$$

Let d be the penetration depth of the thermal wave; we can write

$$x = d, \quad T_2|_{x=d} = T_i \tag{11}$$

$$\frac{\partial T_2}{\partial x}\bigg|_{x=d} = 0. \tag{12}$$

The initial conditions are

$$T_2(x, t=0) = T_i \tag{13}$$

$$s(t=0) = 0 \tag{14}$$

$$d(t=0) = 0. \tag{15}$$

The accuracy of the solution depends on the choice of the temperature profiles in each phase; we assume

$$T_1 = T_c + A(t)(x-s) + B(t)(x-s)^2 \tag{16}$$

$$T_2 = T_i - (T_i - T_c)\left(\frac{d-x}{d-s}\right)^n. \tag{17}$$

Both equations are compatible with expressions (8), (9), (11), (12) if $n > 1$, and (13).

Suppose the heat equation (5) is valid for the point $x = s$, we obtain

$$2Ba_1 = -A\dot{s}. \tag{18}$$

Boundary condition (10) yields

$$T_0 = -As + Bs^2 + T_c. \tag{19}$$

From equations (18) and (19) we obtain the expressions of A and B as functions of s and $\dot{s}$

$$A(t) = \frac{T_c - T_0}{s\left(1 + \dfrac{s\dot{s}}{2a_1}\right)} \tag{20}$$

$$B(t) = -\frac{\dot{s}(T_c - T_0)}{2a_1 s\left(1 + \dfrac{s\dot{s}}{2a_1}\right)}. \tag{21}$$

Suppose the heat equation for the liquid, equation (6), is valid for the point $x = s$, we obtain a differential equation

$$\dot{s} = \frac{(n-1)a_2}{(d-s)}. \tag{22}$$

Boundary condition (7) at the interface gives another differential equation

$$\frac{\lambda_1 (T_c - T_0)}{s\left(1 + \dfrac{s\dot{s}}{2a_1}\right)} - \lambda_2 \frac{n(T_i - T_c)}{d-s} = \rho_1 L\dot{s}. \tag{23}$$

Eliminating $(d-s)$ between equations (22) and (23) yields

$$\frac{\lambda_1 (T_c - T_0)}{1 + \dfrac{s\dot{s}}{2a_1}} = \left[\rho_1 L + \rho_2 c_2 (T_i - T_c)\frac{n}{n-1}\right] s\dot{s}. \tag{24}$$

Introducing the dimensionless parameters ϕ and Ste, we obtain

$$\frac{a_1}{1 + \dfrac{s\dot{s}}{2a_1}} = \left(\frac{1}{Ste} + \phi\frac{n}{n-1}\right) s\dot{s}. \tag{25}$$

Using the initial condition (14), equation (25) is easily solved. The solution is

$$s(t) = 2K\sqrt{(a_1 t)} \tag{26}$$

with

$$K = \frac{1}{2}\sqrt{\left(2\left(\sqrt{\left(1 + \frac{2}{\dfrac{1}{Ste} + \dfrac{\phi n}{n-1}}\right)} - 1\right)\right)}. \tag{27}$$

We remark that solution (26) is formally identical to the exact solution (1). For the case $\phi = 0$, this result has been previously obtained by Goodman [3] and gives reliable values

$$\frac{|K_G - K_N|}{K_N} < 2\% \quad \text{for} \quad 0 < Ste < 2$$

with

$$K_G = \frac{1}{2}\sqrt{(2(\sqrt{(1 + 2Ste)} - 1))}. \tag{28}$$

For the case $\phi \neq 0$, the choice of the exponent n is of great importance. Expanding $e^{-K^2\alpha^2}$ and $\operatorname{erfc}(K\alpha)$ in series and keeping the first-order terms yields

$$e^{-K^2\alpha^2} = 1 + O(K^2\alpha^2) \tag{29}$$

$$\operatorname{erfc}(K\alpha) = 1 - \frac{2}{\sqrt{\pi}} K\alpha + O(K^2\alpha^2). \tag{30}$$

Substituting equations (29) and (30) into equation (4) and comparing equation (4) with equation (25) gives

$$\frac{n}{n-1} = \frac{1}{K\alpha\sqrt{\pi}} + \frac{2}{\pi}. \tag{31}$$

Substituting equation (31) into equation (27), we obtain an approximate solution

$$s(t) = 2K_A\sqrt{(a_1 t)} \tag{32}$$

with

$$K_A = \frac{1}{2}\sqrt{\left(2\left(\sqrt{\left(1 + \frac{2}{\frac{1}{Ste} + \phi\left[\frac{2}{\pi} + \frac{1}{K_A\alpha\sqrt{\pi}}\right]}\right)} - 1\right)\right)} \tag{33}$$

K_A is easily obtained from the sequence

$$K_{j+1} = \frac{1}{2}\sqrt{\left(2\left(\sqrt{\left(1 + \frac{2}{\frac{1}{Ste} + \phi\left[\frac{2}{\pi} + \frac{1}{K_j\alpha\sqrt{\pi}}\right]}\right)} - 1\right)\right)} \tag{34}$$

with the initial value $K_0 = K_G$ (equation (28)).

3. RESULTS AND DISCUSSIONS

For the particular case $\alpha = 1$, we reported the approximate value K_S given by Solomon [12]

$$K_S = \sqrt{(2Ste)}\{1 - (0.21 + \phi(0.51 - 0.169Ste))(0.5642Ste)^B\}$$

with

$$B = \left(\frac{0.93}{1 + 0.69\phi^{0.7}} - 0.15Ste\right) \tag{35}$$

only valid for $0 < \phi < 1$ and $0 < Ste < 1.5$ and compared it, together with our approximate solution K_A, to the exact solution K_N (Table 1). Equation (34) gives a better accuracy for higher values of ϕ and Ste.

We tested our results on a wider range of parameters ($\phi > 0$ and $0 < Ste < 2$) for three values of α ($\alpha = 1$ (Fig. 1), 0.3 (Fig. 2), 3 (Fig. 3)). The relative error ε_A never exceeds 3% and for large values of ϕ, the approximate solution converges to the exact solution, e.g. for $\alpha = 0.3$, $\phi > 2$, $0 < Ste < 2$, $\varepsilon_A < 0.1\%$.

We compared temperature profiles for a very pessimistic case ($\alpha = 3$, $\phi = 1$, $Ste = 1$, $K_N = 0.5027$, $K_A = 0.5131$, $\varepsilon_A = 2.07\%$) with boundary and initial conditions: $T_c = 0$, $T_0 = -20°C$, $T_i = 20°C$ (Fig. 4). We observed these profiles when the boundary reached the value $s = 0.05$ (thus with different times t_A and t_N). Calculations show that to obtain a correct profile for the liquid phase a very high value of n is needed ($n = 319$) and the value of d is also very large ($d = 3.406$). In reality, T_i is reached, because of the high order of the polynomial, for $d \ \#0.100$ ($T_A(d = 0.100, t_A) = 19.83$ and $T_N(d = 0.100, t_N) = 19.98$) the coincidence

Table 1.

Ste	ϕ = 0	0.1	0.5	1	
	0.1568	0.1552	0.1488	0.1415	K_N
0.05	0.38	0.38	0.27	0.28	ε_A (%)
	0	−0.39	−1.34	−0.21	ε_S (%)
	0.2200	0.2166	0.2037	0.1891	K_N
0.1	0.70	0.69	0.59	0.48	ε_A (%)
	0.04	−0.55	−1.72	−0.48	ε_S (%)
	0.4648	0.4461	0.3834	0.3246	K_N
0.5	2.08	1.90	1.46	1.05	ε_A (%)
	−0.11	−1.01	−3.86	−0.37	ε_S (%)
	0.6200	0.5827	0.4698	0.3778	K_N
1	2.42	2.21	1.62	1.16	ε_A (%)
	0	−1.11	−7.53	−5.13	ε_S (%)

$\varepsilon_A = (K_N - K_A)/K_N$, $\varepsilon_S = (K_N - K_S)/K_N$.

Comparison between approximate solution (equation (34)) and Neumann's solution.

Comparison between Solomon's approximation [12] and Neumann's solution.

For $0 < Ste < 1$, $\alpha = 1$, $0 < \phi < 1$

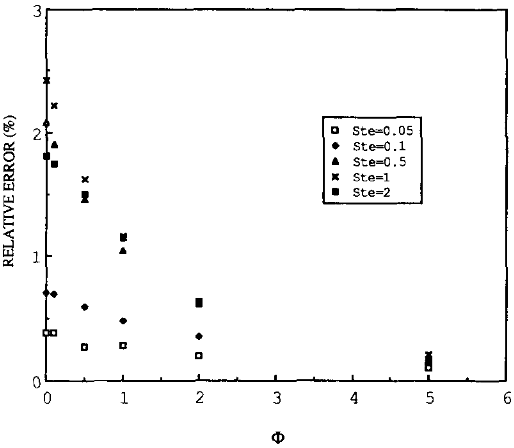

FIG. 1. Relative error $\varepsilon_A = (K_A - K_N)/K_N$ vs ϕ for $\alpha = 1$, $0 < \phi < 6$, $0 < Ste < 2$.

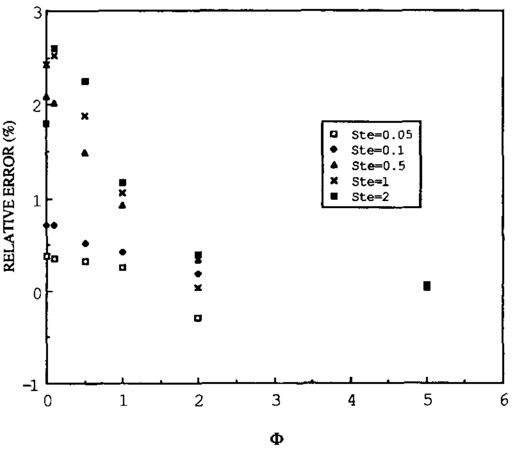

FIG. 2. Relative error $\varepsilon_A = (K_A - K_N)/K_N$ vs ϕ for $\alpha = 0.3$, $0 < \phi < 6$, $0 < Ste < 2$.

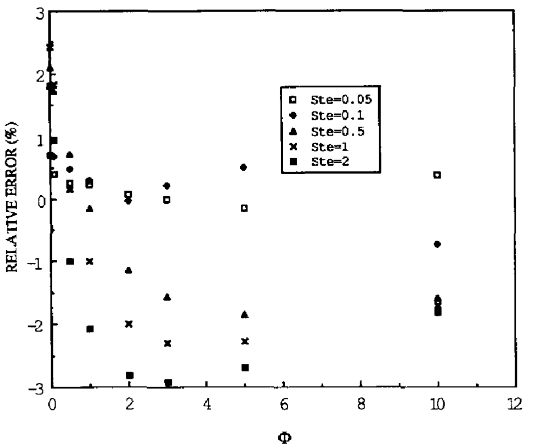

FIG. 3. Relative error $\varepsilon_A = (K_A - K_N)/K_N$ vs ϕ for $\alpha = 3$, $0 < \phi < 10$, $0 < Ste < 2$.

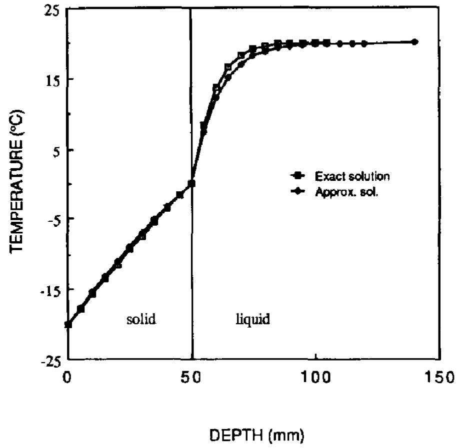

FIG. 4. Comparison between exact and approximate temperature profiles for $\alpha = 3$, $\phi = 1$, $Ste = 1$, $K_A = 0.5131$, $K_N = 0.5027$.

between the approximate and the exact temperature profiles is quite good, especially in the solid region (Fig. 4).

4. CONCLUSION

Using polynomial approximations for the temperature profiles in the solid and liquid phases, equation (34) yields a very precise approximate solution of Neumann's problem. The constant K_A is calculated with an accuracy better than 3% for $0 < Ste < 2$, $0.3 < \alpha < 3$ and $\phi > 0$. For large values of ϕ, the convergence is excellent. The temperature profiles obtained with this method are also very satisfactory.

All calculations can be made very easily on a pocket calculator and should provide fairly good approximations for engineering calculations.

REFERENCES

1. H. Carslaw and J. Jaeger, *Conduction of Heat in Solids*, 2nd Edn. Clarendon Press, London (1959).
2. A. L. London and R. A. Seban, Rate of ice formation, *Trans. Am. Soc. Mech. Engrs* **64**, 105 (1943).
3. T. R. Goodman, The heat balance integral and its application to problems involving a change of phase, *J. Heat Transfer* **80**(2), 335–342 (1958).
4. F. Megerlin, Geometrisch eindimensionale Wärmeleitung beim Schmelzen und Erstarren, *Forsch. IngWes.* **34**(2), 40–46 (1968).
5. G. E. Bell, Solidification of a liquid about a cylindrical pipe, *Int. J. Heat Mass Transfer* **22**, 1681–1686 (1979).
6. R. I. Pedroso and G. A. Domoto, Perturbation solutions for spherical solidification of saturated liquids, *J. Heat Transfer* 42–46 (February 1973).
7. L. N. Gutman, On the problem of heat transfer in phase-change materials for small Stefan numbers, *Int. J. Heat Mass Transfer* **29**(6), 921–926 (1986).
8. J. P. Gupta, An approximate method for calculating the freezing outside spheres and cylinders, *Chem. Engng Sci.* **28**, 1629–1633 (1973).
9. V. J. Lunardini, Phase change around a circular cylinder, *J. Heat Transfer* **103**, 598–600 (1981).
10. W. W. Yuen, Application of the heat-balance integral to melting problems with initial subcooling, *Int. J. Heat Mass Transfer* **23**, 1157–1160 (1980).
11. A. M. C. Chan, P. Smereka and M. Shoukri, An approximate analytical solution to the freezing problem subject to convective cooling and with arbitrary initial liquid temperatures, *Int. J. Heat Mass Transfer* **26**, 1712–1715 (1983).
12. A. D. Solomon, An easily computable solution to a two-phase Stefan problem, *Solar Energy* **23**, 525–528 (1979).